Suggested Algorithm for Tracking Particles With Multiple rf Systems

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We define a central orbit with nominal energy E_0 and a corresponding nominal revolution period T_0.

At each step of the tracking, we follow the motion for T_0 seconds. During this fixed time period a particle may pass through the cavities 0, 1, or 2 times (once being the normal case).

Initial Values

At t=0, the following quantities are initialized:

The time of passage of the previous particle through the cavities:

$$p = 0$$

Beam loading voltage at time=p for cavity mode $k, \quad 1 \le k \le m$

$$U_k^{(0)} = 0$$

and the derivative of the beam loading voltage

$$U_k^{(1)} = 0$$

Non-zero values may be used if one wants to take in to account some history for times less than zero. Setting m=0 means that beam loading is to be ignored and that initial values are not required.

The trajectory of particle i is described by the coordinates

$$\theta_i = \text{azimuthal position}, \; 0 \le \theta_i < 2\pi$$

$$E_i + E_0 = \text{particle energy}$$

The angular revolution frequency is derived from the energy

$$\begin{aligned}\omega_i &= \Omega(E_i) \\ &= \frac{2\pi}{T_0}\left(1+\eta\frac{E_i}{E_0}+\ldots\right)\end{aligned}$$

where the ... indicates that higher order terms may be included if desired.

The charge per particle is:

q_0=Number of particles in the beam/number of particles simulated

Integration of particle motion

The following steps describe the procedure for updating the coordinates of each particle for the period of time $(n-1)T_0 < t \le nT_0$. Equations that have the same quantities on the left and right hand sides are to be interpreted as "replacements statements."

1. For all i, $1 \le i \le Ni$ compute the time τ_i to arrive at the rf cavities:

$$\tau_i = \frac{\theta_i}{\omega_i}$$

$$t_i = 0$$

2. Steps *a* and *b* are iterated twice to account for particles that may pass thru the rf cavities twice.

a. Order particles such that for all i

$$\tau_i < \tau_{i+1}$$

b. Process in the time order determined in step 2*a*. The ordering is required to get the proper causal calculation of the beam loading voltage.

Define t_i as the time for which the particle has been tracked so far

Update t_i to the time of passage through the rf cavity

$$t_i = \tau_i$$

Calculate the energy after passage thru the rf cavity

$$E_i = E_i + \sum_{j=1}^{n_r} V_j\left[(n-1)T_0 + t_i\right] + \sum_{k=1}^{m}\left\{W_k + U_k\left[\Delta t_i, U_k^{(0)}, U_k^{(1)}\right]\right\}$$

where n_r is the number of rf systems, V_j is the rf voltage, W_k is the energy lost to mode k from the particle passing through the cavity, and U_k is the beam loading voltage for mode k.

The rf voltage is assumed to be given by amplitude and phase modulation of the rf frequency $\omega_0 = 2\omega / T_0$

$$V_j\left(nT_0 + t_i\right) = \left[A_{0j} + \delta A_j\left(nT_0 + t_i\right)\right]\sin\left[\frac{2\pi h t_i}{T_0} + \varphi_j\left(nT_0 + t_i\right)\right]$$

where δA, and ϕ_j are arbitrary functions (see Appendix I).

The energy lost to mode k for the cavities is given by the formula

$$W_k = \alpha_k q_0 R_k \omega_{rk}$$

The quantities appearing in the above formula are properties of the cavity mode and are defined in Appendix II.

Δt_i is the length of time since the passage of the last particle:

$$\Delta t_i = \begin{Bmatrix} t_i - p & i \neq 1 \\ t_i + T_0 - p & i = 1 \end{Bmatrix}$$

The formula for the beam loading voltage from all particles that have previously passed thru the cavity is (see Appendix II)

$$U_k\left(\Delta t_i\right) = e^{-\alpha_k \omega_{rk} \Delta t_i}\left[U_k^{(0)} \cos\sqrt{1-\alpha_k^2}\,\omega_{rk}\Delta t_i + \left(U_k^{(1)}/\omega_{rk} + \alpha U_k^{(0)}\right)\frac{\sin\sqrt{1-\alpha_k^2}\,\omega_{rk} t}{\sqrt{1-\alpha_k^2}}\right]$$

Update the time of passage of the previous particle

$$p = t_i$$

Update the beam loading voltage and its derivative to account for the passage of the current particle:

$$U_k^{(0)} = U_k\left[\Delta t_i, U_k^{(0)}, U_k^{(1)}\right] + 2\alpha q_0 R_k \omega_{rk}$$

$$U_k^{(1)} = \frac{\partial U_k}{\partial \Delta t_i}\left[\Delta t_i, U_k^{(0)}, U_k^{(1)}\right] - 4\alpha^2 q_0 R \omega_r^2$$

Update the revolution frequency:

$$\omega_i' = \Omega\left(E_i'\right)$$

Update the time until the next pass through the rf cavity

$$\tau_i = \frac{2\pi}{\omega_i} + t_i$$

3. After all particles have been processed, step 2 is repeated with the new τ_i's.

4. Update the azimuthal position

$$\theta_i = \omega_i\left(T_0 - t_i\right)$$

5. Done. All coordinates have been updated to the time T_0.

Appendix I

We assume that the voltage is given by amplitude and phase modulation of the nominal rf signal as follows

$$V(t) = [A_0 + \delta A(t)]\sin\left[\frac{2\pi ht}{T_0} + \varphi(t)\right] \qquad [A1.1]$$

$\delta A(t)$ and $\varphi(t)$ are arbitrary functions that will generally change very slowly over the time scale T_0. $\delta A(t)$ and $\varphi(t)$ can be parameterized in a number of ways - polynominial functions could be a good choice. The low order terms of $\varphi(t)$ have a special significance.

$\varphi(0)$ is the initial phase offset - determines which particles are intially in the rf bucket

$\frac{d\varphi}{dt}$ is the frequency offset, proportional to the energy offset

$\frac{d^2\varphi}{dt^2}$ is proportional to the rate of acceleration

Higher order terms describe the rate of change of acceleration with time

The reason for choosing to integrate in time steps of T_0 is that the argument of the sine function is always reasonable - perhaps a few 1000's.

$$V(nT_0 + t_i) = [A_0 + \delta A(nT_0 + t_i)]\sin\left[\frac{2\pi ht_i}{T_0} + \varphi(nT_0 + t_i)\right] \qquad [A1.2]$$

Of course, one could use smaller or larger multiples of the rf period. Smaller multiples of the rf period, say (h-a few)*T0/h could eliminate the need to check for particles that pass through the rf cavities twice in one integration period.

Appendix II

Rf cavities are frequently described by an equivalent lumped-element circuit consisting of the parallel combination of an inductor, a resistor, and a capacitor. The circuit equation for this system is

$$Q_0 = Q_L + Q_R + Q_C \qquad \text{[A2.1]}$$

where Q_0 is the charge deposited by an external current source (the beam), and Q_L, Q_R, and Q_C are the charge that has passed thru the inductive, resistive, and capacitive elements respectively.

$$\frac{1}{C}\frac{d^2Q_c}{dt^2} + \frac{1}{RC^2}\frac{dQ_c}{dt^2} + \frac{1}{LC}Q_c = \frac{1}{C}\frac{d^2Q_0}{dt^2} \qquad \text{[A2.2]}$$

From equations A2.1 thru A2.2 one can derive the differential equation

$$\frac{1}{C}\frac{d^2Q_c}{dt^2} + \frac{1}{RC^2}\frac{dQ_c}{dt} + \frac{1}{LC}Q_c = \frac{1}{C}\frac{d^2Q_0}{dt^2} \qquad \text{[A2.3]}$$

Recognizing that the voltage is given by *V*=*Q*/*C* and using

$$\begin{aligned} \omega_r &= \frac{1}{\sqrt{LC}} \\ Q &= \frac{R}{\omega_r L} \\ L &= \frac{R}{\omega_r Q} \\ C &= \frac{Q}{R\omega_r} \end{aligned} \qquad \text{[A2.4]}$$

$$\frac{d^2V}{dt^2} + \frac{\omega_r}{Q}\frac{dV}{dt} + \omega_r^2 V = \frac{R\omega_r}{Q}\frac{d^2Q_0}{dt^2} \qquad \text{[A2.5]}$$

Solving A2.4 by the Laplace transform method with the condition $Q_0(0) = Q_0'(0) = 0$ one obtains

$$V(t) = \int_{a-i\infty}^{a+i\infty} \frac{V_0' + (s + \omega_r/Q)V_0 + s^2\tilde{Q}(s)R\omega_r/Q}{s^2 + s\omega_r/Q + \omega_r^2} e^{st} ds \qquad \text{[A2.5]}$$

Specializing to the passage of a particle of charge q_0

$$Q_0(t) = \begin{Bmatrix} 0 & t \le 0 \\ q_0 & t > 0 \end{Bmatrix} \qquad [A2.6]$$

$$\tilde{Q}(s) = \frac{q_0}{s} \qquad [A2.7]$$

$$V(t) = \int_{a-i\infty}^{a+i\infty} \frac{V_0' + V_0\,\omega_r/Q + s\left(V_0 + q_0 R\omega_r/Q\right)}{s^2 + s\omega_r/Q + \omega_r^2} e^{st} ds \qquad [A2.8]$$

We define s_1 and s_2 to be the roots of the equiation

$$s^2 + \frac{s\omega_r}{Q} + \omega_r^2 = 0 \qquad [A2.9]$$

$$s_1 = \left(i\sqrt{1-\alpha^2} - \alpha\right)\omega_r \qquad [A2.10]$$

$$s_2 = \left(-i\sqrt{1-\alpha^2} - \alpha\right)\omega_r \qquad [A2.11]$$

where

$$\alpha = \frac{1}{2Q} \qquad [A2.12]$$

The inverse Laplace transform is easily found by decomposing the denominator into partial fractions as follows:

$$\frac{s}{(s-s_1)(s-s_2)} = \frac{1}{(s_1-s_2)}\left(\frac{s_1}{s-s_1} - \frac{s_2}{s-s_2}\right) \qquad [A2.13]$$

$$\frac{1}{(s-s_1)(s-s_2)} = \frac{1}{(s_1-s_2)}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right) \qquad [A2.14]$$

One finally obtains

$$\begin{aligned}
V(t) &= \frac{1}{(s_1-s_2)}\left[\left(V_0' + V_0\,\omega_r/Q\right)\left(e^{s_1 t} - e^{s_2 t}\right) + \left(V_0 + q_0 R\omega_r/Q\right)\left(s_1 e^{s_1 t} - s_2 e^{s_2 t}\right)\right] \\
&= e^{-\alpha\omega_r t}\left[\left(V_0'/\omega_r + V_0/Q\right)\frac{\sin\sqrt{1-\alpha^2}\,\omega_r t}{\sqrt{1-\alpha^2}\,\omega_r} + \left(V_0 + q_0 R\omega_r/Q\right)\left(\cos\sqrt{1-\alpha^2}\,\omega_r t - \frac{\alpha}{\sqrt{1-\alpha^2}}\sin\sqrt{1-\alpha^2}\,\omega_r t\right)\right] \\
&= e^{-\alpha\omega_r t}\left[\left(V_0 + 2\alpha q_0 R\omega_r\right)\cos\sqrt{1-\alpha^2}\,\omega_r t + \left(V_0'\ \omega_{r\,0} + \alpha V_0 - 2\alpha^2 q_0 R\omega_r\right)\frac{\sin\sqrt{1-\alpha^2}\,\omega_r t}{\sqrt{1-\alpha^2}}\right]
\end{aligned}$$

Note that this expression is also valid for $t<0$, provided q0 is set to 0.